

Some $O(\alpha^2)$ annihilation type contributions to the orthopositronium width

V. Antonelli¹, V. Ivanchenko², E. Kuraev³, V. Lalienna¹

¹ Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland

² Institute of Nuclear Physics, Siberian Division, 630090 Novosibirsk, Russia

³ JINR, 141980 Dubna, Russia

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Abstract. We consider some radiative corrections to the lowest order annihilation diagram for the orthopositronium decay rate. The insertion of the renormalized vertex correction in the annihilation graph gives $1.6278 (\alpha/\pi)^2 \Gamma_0$. We compute also the contribution of the square of the lowest order annihilation amplitude, which turns out to be $0.1702 (\alpha/\pi)^2 \Gamma_0$. Finally, we obtain a term $\alpha^2 \ln \alpha \Gamma_0$ arising from the correction to the light–light scattering block due to the exchange of one coulombic photon, in agreement with earlier computations.

1 Introduction

Measurements of orthopositronium (O_{ps}) decay rate in the recent years pose a great challenge in QED due to the large discrepancy between the experimental result and the theoretical predictions [1]. In fact, the two most accurate experimental rates [2,3]

$$\lambda_{O_{ps}}^{exp} = 7.0514 \pm 0.0014 \mu s^{-1} \quad \text{and} \\ \lambda_{O_{ps}}^{exp} = 7.0482 \pm 0.0016 \mu s^{-1},$$

deviate by 9.4σ and 6.2σ from the theoretical value¹, whose most accurate estimate to order α/π has been given in [5]:

$$\lambda_{O_{ps}}^{QED}(PS \rightarrow 3\gamma) = 7.038236 \pm 0.000010 \mu s^{-1}.$$

The decay rate to leading order,

$$\Gamma_0 = \frac{\alpha^6 m c^2}{\hbar} \frac{2(\pi^2 - 9)}{9\pi} = 7.21117 \mu s^{-1}, \quad (1)$$

was computed by Ore and Powell [6] and the order α corrections by several authors [5,7–10].

Faced up to this difficulty, theorists have made a great effort to compute the next correction, $(\alpha/\pi)^2$, of the perturbative expansion [11–13]:

$$\lambda_{O_{ps}}^{QED}(PS \rightarrow 3\gamma) = \Gamma_0 \left[1 + (-10.2866 \pm 0.0006) \frac{\alpha}{\pi} \right]$$

¹ Recently a new experiment [4] gave the value $7.0398 \pm 0.0025 \pm 0.0015 \mu s^{-1}$ (the first error is statistical and the second systematic), in good agreement with the theoretical expectation. However, an independent confirmation of this measurement is necessary before concluding that the O_{ps} problem is experimental instead of theoretical

$$+ C \left(\frac{\alpha}{\pi} \right)^2 + O((\alpha/\pi)^3) \Big]. \quad (2)$$

To get rid of the theoretical–experimental discrepancy, the coefficient C must be of order 250 ± 40 . Such a large value cannot be excluded, even if it may appear unnatural in the framework of perturbation theory (PT). If the results of [4] are correct (see footnote 1) only $C \approx 30$ is required.

The computation of C is very hard due to the large number of Feynman diagrams contributing to the α^2 order of PT. Some of these have been already calculated: the vacuum polarization type corrections to the first order graphs were considered in [11], the radiative corrections to the light–light scattering block in [12], and the square of the first order amplitude in [13]. The relativistic corrections, i.e. those associated with the expansion in $v/c \sim \alpha$, where v is the relative velocity of the e^+e^- pair in positronium, were taken into account up to order α^2 in [14].

All the contributions to the amplitude up to second order PT may be written in the form:

$$M = M_0 + \frac{\alpha}{\pi} (M_{B'} + M_A + M_1) \\ + \left(\frac{\alpha}{\pi} \right)^2 (M_{AB'} + M_{AR} + M_2) + O(\alpha^3), \quad (3)$$

where² M_1 represents the sum of all the first order amplitudes with the exceptions of the annihilation diagram

² Note that here and in the rest of the paper we write explicitly the powers of α/π , relative to the lowest order M_0 , for each amplitude. (For example, M_A is of the same order as M_0). On the contrary, we omit them in the text. Note also that we will not write any power of α/π for the two unsubtracted binding amplitudes, M_B and M_{AB} , since they contain terms of different order in α/π

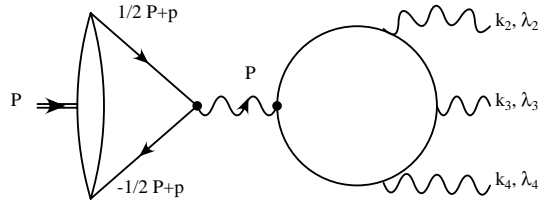


Fig. 1. The annihilation graph

(see Fig. 1), denoted by M_A , and the subtracted binding amplitude, $M_{B'}$ [5]. The second order annihilation type corrections are given by the subtracted binding diagram, $M_{AB'}$ (Fig. 2a) and the radiative corrections to the light–light scattering block, M_{AR} (an example of which is given in Fig. 2b). M_2 denotes the remaining (non–annihilation type) second order amplitudes.

In this paper we consider the second order corrections given by M_A and $M_{AB'}$ and the logarithmic enhancement produced by a coulombic one photon exchange in M_{AR} . The contribution of these corrections to the decay rate has the form

$$\left(\frac{\alpha}{\pi}\right)^2 \left[2 \operatorname{Re} (M_{AB'}^* M_0) + |M_A|^2 \right] + \alpha^2 \ln \alpha |M_0|^2 . \quad (4)$$

The previous expression must be summed over the final photon polarizations, averaged over the O_{ps} spin states and integrated over the phase space of the three final photons, with the proper kinematical factors (see for example [5]).

The remaining part of the paper is organized as follows: in the second section we compute the contributions of the binding corrections to the lowest order annihilation diagram. In section three, using the known results for the light–light scattering tensor, we compute the contribution of the square of the lowest order annihilation amplitude. Finally, in the fourth section we consider the exchange of one coulombic photon in the light–light scattering block of the annihilation amplitude, finding a logarithmic contribution in agreement with earlier calculations [9,15,12,16].

2 Radiative corrections to the annihilation process

It is well known [17,18] that the contribution to the amplitude for the O_{ps} decay rate originated by the binding diagram contains also the lowest order approximation:

$$M_B = M_0 + \frac{\alpha}{\pi} M_{B'} . \quad (5)$$

Therefore only the subtracted binding diagram $M_{B'}$ must be included, otherwise M_0 would be counted twice. This phenomenon occurs due to the presence of the coulombic part in the virtual photon propagator, which had been already taken into account when solving the Bethe–Salpeter equation for the O_{ps} wave function.

It is quite clear that the analogous phenomenon should come out in the “binding” type radiative correction to the

lowest order annihilation diagram (see Fig. 2a), and in fact we shall show in this section that the amplitude M_A is contained in M_{AB} . Therefore, we can write:

$$M_{AB} = \frac{\alpha}{\pi} M_A + \left(\frac{\alpha}{\pi}\right)^2 M_{AB'} , \quad (6)$$

where $M_{AB'}$ is the subtracted binding–annihilation amplitude.

We express the amplitude as a product

$$M_{AB}^{(m,\lambda)} = \frac{-i}{4m^2} T_\rho^{(m)\rho} G_\rho^{(\lambda)} . \quad (7)$$

In the previous formula the 4–vector $G_\rho^{(\lambda)}$ describes the transition of the heavy photon to three real ones, $\lambda = (\lambda_2, \lambda_3, \lambda_4)$ stands for the set of the three polarizations of the final photons, $\lambda_i = \pm 1$, and $T_\rho^{(m)}$ is the order α correction to the annihilation current 4–vector of the positronium in the polarization state ϵ_m . Explicitly:

$$\begin{aligned} & T_\rho^{(m)} \\ &= -\frac{i e \alpha}{4\pi} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{i\pi^2} \frac{\Delta_{\mu\nu}(k-p)}{(k-p)^2} \times \\ & \frac{\operatorname{Tr} \left\{ \Psi^{(m)}(p) \gamma_\mu \left[-\hat{P}/2 + \hat{k} + m \right] \gamma_\rho \left[\hat{P}/2 + \hat{k} + m \right] \gamma_\nu \right\}}{[(-P/2 + k)^2 - m^2][(P/2 + k)^2 - m^2]} \\ & + C_t , \end{aligned} \quad (8)$$

where C_t stands for the contribution of the vertex counterterm. We use the notation $\hat{k} = k_\mu \gamma^\mu$ and here and everywhere in this paper a term $i0$ must be implicitly understood in each factor of the denominator arising from a propagator (only in (56) the $i0$ will be explicitly written). In (8) P is the O_{ps} momentum in its rest frame:

$$P = (2W, 0, 0, 0), \quad \text{with } W \approx m - \frac{\gamma^2}{2m}, \quad \text{and } \gamma = \frac{m\alpha}{2} .$$

The wave function $\Psi^{(m)}(p)$ of O_{ps} relevant to our approximation is (see for example [5,19]):

$$\Psi^{(m)}(p) = (2\pi) \delta(p_0) \sqrt{2m} \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}_m \\ 0 & 0 \end{bmatrix} \Phi(\mathbf{p}), \quad (9)$$

with $\Phi(\mathbf{p})$ the nonrelativistic ground–state wave function

$$\Phi(\mathbf{p}) = \phi_0 \frac{8\pi\gamma}{(|\mathbf{p}|^2 + \gamma^2)^2} \quad (10)$$

and the constant ϕ_0 is the wave function at the origin, $\phi_0 = \sqrt{\gamma^3/\pi}$.

The $\Delta_{\mu\nu}$ tensor obviously depends on the gauge we use. The choice of the gauge is subtle when dealing with bound state problems. It has been discussed elsewhere (see for example [10]) that the Coulomb gauge is the most natural for calculations in positronium. However, covariant gauges are simpler for computing radiative corrections, and, among them, the Fried–Yennie (FY) gauge is the most convenient, due to its good infrared behaviour. We shall compute $T_\rho^{(m)}$ both in the FY gauge and in the Coulomb gauge. As expected, the result is the same in both cases, and no gauge correction term must be added when using the FY gauge.

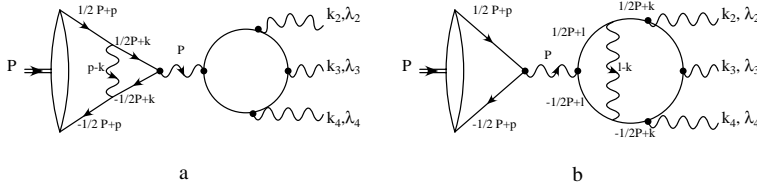


Fig. 2. Two different kinds of corrections to the annihilation graph. **a** is the vertex correction, **b** represents the insertion of a photon into the light–light scattering block, which generates a logarithmically enhanced contribution. The direction of the fermions in the final loop is clockwise for all the two diagrams

2.1 Fried-Yennie gauge

The FY gauge is a covariant gauge defined by

$$\Delta_{\mu\nu}(k) = g_{\mu\nu} + 2 \frac{k_\mu k_\nu}{k^2}. \quad (11)$$

It has good infrared properties, allowing us to work safely with a zero fictitious photon mass from the beginning.

Following [5] we separate the trace entering the integrand of (8) into two pieces, one which remains non-singular at $k=0$ and one containing the contribution of the coulombic photon. To this end, we define

$$\begin{aligned} tr_{\mu\nu\rho}(k) = & Tr \left\{ \Psi^{(m)}(p) \gamma_\mu \left[-\hat{P}/2 + \hat{k} + m \right] \right. \\ & \left. \times \gamma_\rho \left[\hat{P}/2 + \hat{k} + m \right] \gamma_\nu \right\} \end{aligned} \quad (12)$$

and write: $tr_{\mu\nu\rho}(k) = tr_{\mu\nu\rho}(0) + \{tr_{\mu\nu\rho}(k) - tr_{\mu\nu\rho}(0)\}$.

We consider first the term $tr_{\mu\nu\rho}(0)$. The γ matrices algebra leads to:

$$\begin{aligned} tr_{\mu\nu\rho}(0) = & -4W^2 \delta_{\mu 0} \delta_{\nu 0} Tr \left\{ \gamma_\rho \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}_m \\ 0 & 0 \end{bmatrix} \right\} \\ & + O(\alpha^2). \end{aligned} \quad (13)$$

To arrive to the last expression we have used the fact that

$$\begin{aligned} \left(\frac{1}{2} \hat{P} - m \right) \Psi^{(m)}(p) &= O(\alpha^2) \\ \Psi^{(m)}(p) \left(\frac{1}{2} \hat{P} + m \right) &= O(\alpha^2). \end{aligned} \quad (14)$$

The integration in p_0 in (8) is trivial, using the delta function entering the formula (9). It remains the following integral:

$$\begin{aligned} & \int \frac{d^3 p}{(2\pi)^3} \frac{8\pi\gamma}{(|\mathbf{p}|^2 + \gamma^2)^2} \int \frac{d^4 k}{(i\pi^2)} \\ & \times \frac{-W^2 \left(1 + 2 \frac{k_0^2}{(k-p)^2} \right)}{(k-p)^2 \left[(k + \frac{1}{2}P)^2 - m^2 \right] \left[(k - \frac{1}{2}P)^2 - m^2 \right]}, \end{aligned} \quad (15)$$

whose result, $\frac{\pi}{\alpha} - 3 + O(\alpha^2)$, can be found in [5]. Hence the contribution of this term to $T_\rho^{(m)}$ is:

$$-i \frac{\alpha^3 m^2}{\pi} Tr \left\{ \gamma_\rho \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}_m \\ 0 & 0 \end{bmatrix} \right\} \left(\frac{\pi}{\alpha} - 3 \right). \quad (16)$$

Let us now consider the remaining term, $tr_{\mu\nu\rho}(k) - tr_{\mu\nu\rho}(0)$. In this case the integral is free from infrared singularities and we can put $p = 0$ in the loop integral, introducing an error of order α^2 . However, ultraviolet divergences are present; we regulate them by using dimensional

regularization (the analogous calculation in cut-off regularization is performed in Appendix A). It is important to remember that some care is necessary when using dimensional regularization in the FY gauge. As was shown by G. Adkins [20], it is convenient to choose the tensor $\Delta_{\mu\nu}$ as:

$$\Delta_{\mu\nu}(k) = g_{\mu\nu} + \frac{2}{1-2\epsilon} \frac{k_\mu k_\nu}{k^2}, \quad (17)$$

where $\epsilon = (4-d)/2$ and d is the complex space-time dimension.

Using γ matrices algebra and (14) one can see that the only contribution of $tr_{\mu\nu\rho}(k) - tr_{\mu\nu\rho}(0)$ to the integral (8) is given by:

$$Tr \left\{ \gamma_\mu \hat{k} \gamma_\rho \hat{k} \gamma_\nu \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}_m \\ 0 & 0 \end{bmatrix} \right\}. \quad (18)$$

In this way we arrive to the following integral:

$$\begin{aligned} & - \frac{i e \alpha}{4\pi} \int \frac{d^d k}{i\pi^2 (2\pi\mu)^{-2\epsilon}} \\ & \times \frac{\gamma_\mu \hat{k} \gamma_\rho \hat{k} \gamma_\nu \Delta_{\mu\nu}(k)}{k^2 \left[(k + \frac{1}{2}P)^2 - m^2 \right] \left[(k - \frac{1}{2}P)^2 - m^2 \right]}, \end{aligned} \quad (19)$$

where μ is the dimensional parameter introduced in dimensional regularization. The integral (19) can be evaluated by standard techniques, giving

$$- \frac{i e \alpha}{4\pi} (3D + 8 - 2\delta_{\rho 0}) \gamma_\rho, \quad (20)$$

where $D = \frac{1}{\epsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{m^2}$. (The number $\gamma_E = 0.57721$ is the Euler constant).

Note that current conservation implies $P^\rho G_\rho^{(\lambda)} = 0$. Since we work in the positronium rest frame ($\mathbf{P} = 0$), it follows $G_0^{(\lambda)} = 0$ and the term with $\delta_{\rho 0}$ in (20) can be ignored. The contribution of the remaining term to $T_\rho^{(m)}$ is then:

$$- \frac{i\alpha^3}{4\pi} m^2 (3D + 8) Tr \left\{ \gamma_\rho \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}_m \\ 0 & 0 \end{bmatrix} \right\}. \quad (21)$$

Adding (16) and (21), inserting the result in (7) and noting that the first order annihilation amplitude M_A can be written as

$$\frac{\alpha}{\pi} M_A = - \frac{\alpha^2}{4} Tr \left\{ \gamma_\rho \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}_m \\ 0 & 0 \end{bmatrix} \right\} G^{(\lambda)\rho}(k_2, k_3, k_4), \quad (22)$$

we obtain the following contribution to M_{AB} :

$$\frac{\alpha}{\pi} M_A \left[1 + \frac{\alpha}{\pi} \left(-3 + \frac{3D + 8}{4} \right) \right]. \quad (23)$$

The contribution of the vertex counterterm will cancel the divergence appearing in (23). Due to the Ward identity, this counterterm can be obtained from the self-energy correction to the electron propagator.

To order α the mass operator has the form:

$$\Sigma(l) = \frac{\alpha}{4\pi} \int \frac{d^4k}{i\pi^2} \frac{\gamma_\mu (\hat{l} - \hat{k} + m) \gamma_\nu \Delta_{\mu\nu}(k)}{k^2 [(l-k)^2 - m^2]}. \quad (24)$$

On mass-shell renormalization conditions imply

$$\frac{i}{\hat{p} - m_0 - \Sigma(p)} \rightarrow \frac{iZ}{\hat{p} - m}, \quad \hat{p} \rightarrow m, \quad (25)$$

where m_0 is the bare electron mass. From (24) it is possible to obtain the electron wave function renormalization constant, which in the FY gauge is [20]:

$$Z_{FY} = 1 - \frac{\alpha}{4\pi} (3D + 4). \quad (26)$$

It follows that the contribution to the amplitude of the vertex counterterm is:

$$-\left(\frac{\alpha}{\pi}\right)^2 \frac{3D+4}{4} M_A. \quad (27)$$

Finally, summing up all the contributions considered here (the terms $tr_{\mu\nu\rho}(0)$ and $tr_{\mu\nu\rho}(k) - tr_{\mu\nu\rho}(0)$, and the counterterm insertion) we get:

$$M_{AB} = \left(1 - 2\frac{\alpha}{\pi}\right) \frac{\alpha}{\pi} M_A. \quad (28)$$

2.2 Coulomb gauge

We will show now that the same result is obtained working in the Coulomb gauge. This gauge is obtained by making the substitution:

$$\frac{-i}{(k-p)^2} \Delta_{\mu\nu}(k-p) \rightarrow G_{\mu\nu}(k-p), \quad (29)$$

where $G_{\mu\nu}^{cb}(q)$ is the Coulomb propagator:

$$\begin{aligned} G_{00}(q) &= \frac{i}{\mathbf{q}^2}, \\ G_{0i}(q) &= G_{i0}^{cb}(q) = 0, \\ G_{ij}(q) &= \frac{i}{q^2} \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2} \right). \end{aligned} \quad (30)$$

Again, using γ matrices algebra and (14), we rewrite the trace in the numerator of (8) as

$$\begin{aligned} tr_{\mu\nu\rho}(k) &= -P_\mu P_\nu Tr [\Psi^{(m)}(p) \gamma_\rho] \\ &+ Tr \left[\Psi^{(m)}(p) \gamma_\nu \hat{k} \gamma_\rho \hat{k} \gamma_\mu \right] \\ &+ Tr \left[\Psi^{(m)}(p) \left(P_\mu \gamma_\nu \hat{k} \gamma_\rho - P_\nu \gamma_\rho \hat{k} \gamma_\mu \right) \right]. \end{aligned} \quad (31)$$

It is easy to see that the last term vanishes after contracting the Lorentz indices with the ones of the photon propagator. The contribution of the first term to $T_\rho^{(m)}$ is

$$-\alpha^3 (16\pi) m^4 Tr \left\{ \gamma_\rho \begin{bmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}_m \\ 0 & 0 \end{bmatrix} \right\} I_0, \quad (32)$$

where I_0 is defined by:

$$\begin{aligned} I_0 &= \int \frac{d^3p}{(2\pi)^3} \frac{8\pi\gamma}{(\mathbf{p}^2 + \gamma^2)^2} \int \frac{d^4k}{(4\pi)^2} \\ &\times \frac{1}{(\mathbf{k} - \mathbf{p})^2 [(-P/2 + k)^2 - m^2] [(P/2 + k)^2 - m^2]}. \end{aligned} \quad (33)$$

This integral has been studied in [18]. Its result is

$$I_0 = \frac{i}{(4\pi)^2} \frac{1}{m^2} \left(\frac{\pi}{\alpha} - 2 \right). \quad (34)$$

Using this result and considering (7) and (22), we find that the contribution of the first term of (31) to the amplitude is

$$\left(1 - 2\alpha/\pi\right) \left(\frac{\alpha}{\pi} M_A\right), \quad (35)$$

which is the total result obtained in the FY gauge.

Now we will show that the contribution of the second term in (31) exactly cancels against the contribution of the Coulomb gauge vertex counterterm. Hence, the result in the Coulomb gauge is the same as in the FY gauge.

The second term of (31) gives raise to an UV divergence. Again, we choose dimensional regularization to give a meaning to the loop integral. We work in $d = 2\omega$ dimensions, with one temporal and $2\omega - 1$ spatial dimensions.

As in the case of the FY gauge, there is no infrared problem for this term, and we can put $p = 0$ in the photon propagator of the integral (8). We have

$$\begin{aligned} e^3 \int \frac{d^{2\omega}k}{(2\pi\mu)^{2\omega}} \\ \times \frac{\gamma_\nu \hat{k} \gamma_\rho \hat{k} \gamma_\mu G^{\mu\nu}(k)}{[(-P/2 + k)^2 - m^2] [(P/2 + k)^2 - m^2]}. \end{aligned} \quad (36)$$

The formulae for integrals of non-covariant functions in this dimensional regularization prescription can be found in [21]. After standard computations and taking into account the fact that the terms with the Lorentz index $\rho = 0$ do not contribute to the amplitude, we obtain:

$$-\frac{i e^3}{(4\pi)^2} \left(\frac{4}{3} D + \frac{20}{9} \right) \gamma_\rho \quad (37)$$

for the case of the temporal propagator, G_{00} , and

$$\frac{i e^3}{(4\pi)^2} \left(\frac{1}{3} D + \frac{20}{9} \right) \gamma_\rho \quad (38)$$

for the contribution of the spatial components G_{ij} of the photon propagator. The total result is therefore

$$\frac{\alpha}{4\pi} D (-i e \gamma_\rho). \quad (39)$$

The contribution of the vertex counterterm is given by $\delta Z_1 (-ie\gamma_\rho)$. Since the vertex counterterm to order α in the Coulomb gauge is [21]

$$\delta Z_1 = -\frac{\alpha}{4\pi} D, \quad (40)$$

we see that the counterterm exactly cancels the contribution of the second term in (31), as we wanted to show, and then, the amplitude is given by (28) also in the Coulomb gauge.

2.3 Conclusion

As claimed at the beginning of this section, we must consider only the subtracted amplitude,

$$M_{AB'} = M_{AB} - \frac{\alpha}{\pi} M_A = -2 \left(\frac{\alpha}{\pi}\right)^2 M_A. \quad (41)$$

Hence the O_{ps} decay rate receives a contribution given by the integral of

$$\frac{1}{3} \sum_m \frac{1}{3!} \sum_\lambda 2 \operatorname{Re} \left(M_{AB'}^{(m\lambda)} * M_0^{(m\lambda)} \right) \quad (42)$$

over the phase space of the three final photons. Since $M_{AB'}$ is proportional to the lowest order annihilation amplitude, the contribution to the width is proportional to the lowest order annihilation width, whose value can be found in [5]: $\Gamma_A = -0.81389 (\alpha/\pi) \Gamma_0$.

Therefore, the renormalized vertex correction to the annihilation amplitude turns out to be:

$$\Gamma_{AB'} = -2 \frac{\alpha}{\pi} \Gamma_A = 1.6278 \frac{\alpha^2}{\pi^2} \Gamma_0. \quad (43)$$

This result exactly coincides with the estimation made by Karshenböim [22]. Notice that, differently from the lowest order annihilation case, it contributes positively in the direction of reducing the theoretical–experimental discrepancy. Its numerical value, however, is too small to make a significant progress.

3 Lowest order annihilation diagram

Now we consider the lowest order annihilation matrix element (Fig. 1):

$$-\frac{\sqrt{4\pi\alpha}}{4m^2} \int \frac{d^4 p}{(2\pi)^4} \operatorname{Tr} \{ \Psi^{(m)}(p) \gamma_\rho \} G^{(\lambda)\rho}. \quad (44)$$

The integral over p is trivially performed. The next stage is to square the resulting matrix element and make the average over the O_{ps} polarization states [5]. After simple calculations, similar to those performed in the paper of one of us [23], – where the production of tree gluonic jets

in electron–positron colliding beams was considered –, we obtain the following contribution to the width³:

$$\Gamma_{A^2} = -m \frac{\alpha^4}{2^{10} 9 \pi^3} \times \int d^3 \nu \delta(\Sigma \nu_i - 2) \Sigma_\lambda G_\rho^{(\lambda)} G^{(\lambda)\rho}. \quad (45)$$

To arrive to (45), we have expressed the phase space volume of the three final photons as:

$$\int \frac{d^3 k_2 d^3 k_3 d^3 k_4}{\omega_2 \omega_3 \omega_4} \delta^4(P - k_2 - k_3 - k_4) = 8\pi^2 m^2 \int d\nu_2 d\nu_3 d\nu_4 \delta(2 - \nu_2 - \nu_3 - \nu_4), \quad (46)$$

where $\nu_i = \frac{\omega_i}{m} = \frac{|\mathbf{k}_i|}{m}$.

At this point we use the results of papers [23,24], namely:

$$-\Sigma_\lambda G_\rho^{(\lambda)} G^{(\lambda)\rho} = 2^6 \alpha^4 [R(234) + R(324) + R(423)] \quad (47)$$

$$R(234) = R(243)$$

$$\begin{aligned} &= \frac{1}{3} \left| \mathcal{E}_{-++}^{(2)}(234) \right|^2 + \left| \mathcal{E}_{+++}^{(2)}(234) \right|^2 \\ &+ \frac{\nu_2}{\nu_3 \nu_4 a_2} \left| \mathcal{E}_{-++}^{(1)}(324) \right|^2 \\ &+ \frac{1}{\nu_2^2} \left| \mathcal{E}_{+++}^{(1)}(234) + \mathcal{E}_{+++}^{(1)}(243) \right|^2 \\ &+ \frac{a_3 a_4}{\nu_2^2 a_2} \left| \frac{1}{a_3} \mathcal{E}_{+++}^{(1)}(234) - \frac{1}{a_4} \mathcal{E}_{+++}^{(1)}(243) \right|^2, \quad (48) \end{aligned}$$

where $a_i = 1 - \nu_i$. The rather cumbersome functions \mathcal{E} , whose arguments are the ν_i , were calculated in the paper of Costantini et al. [24]. Their expressions are explicitly written in Appendix B. After numerical integration over the phase space, we find:

$$\Gamma_{A^2} = b \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0, \quad (49)$$

$$\begin{aligned} b &= \frac{\int d^3 \nu \delta(2 - \Sigma_i \nu_i) [R(234) + R(324) + R(423)]}{32(\pi^2 - 9)} \\ &= 0.17021(10). \quad (50) \end{aligned}$$

We would like to stress that the contribution from muon (and hadrons) as fermions in the loop is negligible (of order $(m_e/m_\mu)^4$) [7].

In conclusion, the total correction found here, adding (50) to (43), is $1.7983 (\alpha/\pi)^2 \Gamma_0$. Manifestly, it is still far from solving the discrepancy between the modern theoretical and experimental results. If the O_{ps} problem is to be solved by this kind of perturbation theory, larger contributions to the width must be searched in another class of diagrams.

³ Remember that the vector $G_\rho^{(\lambda)}$ is space-like, as explained before. Therefore, Γ_{A^2} is positive, in spite of what at first sight might seem due to the minus sign in the r.h.s. of (45)

4 Coulombic exchange in the light-light scattering block

In this section we shall obtain a known logarithmically enhanced contribution arising from radiative corrections to the light–light scattering block. The whole set of these radiative corrections has been already computed by Adkins and Lymberopoulos [12]. We present here a simple computation of the term of order $\alpha^2 \ln \alpha$, which arises from the diagram displayed in Fig. 2b.

The contribution to the amplitude of the radiative corrections to the light–light scattering block can be represented by:

$$M_{AR}^{(\lambda)} = -i e \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \{ \Psi^{(m)}(p) \gamma_\rho \} \times \frac{-i}{4m^2} G_{(R)}^{(\lambda)\rho}(k_2, k_3, k_4), \quad (51)$$

where the subscript (R) in $G_{(R)}^{(\lambda)\rho}$ means the order α radiative corrections.

We are interested in the contributions to $G_{(R)}^{(\lambda)\rho}$ of the loops of the form represented by Fig. 2b.⁴ It is given by

$$- \frac{i e^3}{(4\pi)^2} \int \frac{d^4 l}{(2\pi)^4} \int \frac{d^4 k}{i\pi^2} \frac{\Delta_{\mu\nu}(k-l)}{(l-k)^2} \times \text{Tr} \left\{ \frac{1}{-\frac{1}{2}\hat{P} + \hat{l} - m} \gamma_\nu \frac{1}{-\frac{1}{2}\hat{P} + \hat{k} - m} \right. \\ \left. \times \mathcal{O}_{234}^{(\lambda)} \frac{1}{\frac{1}{2}\hat{P} + \hat{k} - m} \gamma_\mu \frac{1}{\frac{1}{2}\hat{P} + \hat{l} - m} \gamma_\rho \right\}. \quad (52)$$

In the previous formula $\Delta_{\mu\nu}(k-l)$ is chosen in the FY gauge (11) and $\mathcal{O}_{234}^{(\lambda)}$ describes the annihilation of a pair $e^+ e^-$ at rest to three photons, namely:

$$\mathcal{O}_{234}^{(\lambda)} = -i e^3 \gamma \cdot \epsilon_{\lambda_4} \frac{1}{-\frac{1}{2}\hat{P} + \hat{k} + \hat{k}_4 - m} \times \gamma \cdot \epsilon_{\lambda_3} \frac{1}{\frac{1}{2}\hat{P} + \hat{k} - \hat{k}_2 - m} \gamma \cdot \epsilon_{\lambda_2}. \quad (53)$$

It is worthwhile to underline that there are two regions in the loop momenta space from which the integral (52) receives the main contributions: one, where the fermion momenta are far off mass–shell, which correspond to $l, k \sim m$, and another one with fermion momenta almost on mass–shell, $l, k \sim \alpha m$. It is this last region which originates the logarithmic term. In the analysis of this region we can neglect k and l in the numerators of fermion propagators (after rationalizing them) and in $\mathcal{O}_{234}^{(\lambda)}$. Then, using γ matrices algebra, we can rewrite the trace in (52) in the following way:

$$- P_\mu P_\nu m^2 \text{Tr} \{ \mathcal{O}_{234}^{(\lambda)} (1 + \gamma_0) \gamma_\rho (1 - \gamma_0) \} + O(\alpha^2), \quad (54)$$

⁴ Strictly speaking, Fig. 2b represents only a typical loop. There are five more graphs of the same kind corresponding to permutations of the three final photon states

where we have used the fact that $P^2 - 4m^2 = O(\alpha^2)$. The integration in k is now identical to that of (15). We use the result [5]:

$$\int \frac{d^4 k}{i\pi^2} \times \frac{-m^2 \Delta_{00}(k-l)}{(k-l)^2 [(-\frac{1}{2}P+k)^2 - m^2] [(\frac{1}{2}P+k)^2 - m^2]} \approx \frac{\pi m}{|\mathbf{l}|} \arctan \frac{|\mathbf{l}|}{\gamma} - 3. \quad (55)$$

The integral in l can be performed in two steps. First, we integrate l_0 using the method of residues:

$$\int \frac{dl_0}{2\pi} \times \frac{1}{[(-\frac{1}{2}P+l)^2 - m^2 + i0] [(\frac{1}{2}P+l)^2 - m^2 + i0]} \approx \frac{i}{4m} \frac{1}{|\mathbf{l}|^2 + \gamma^2}. \quad (56)$$

The remaining integral over the spatial 3–momentum \mathbf{l} is logarithmically divergent. However, we do not worry here for the ultraviolet sector, which is outside the integration region we are considering and gives no logarithmic contribution in α . The considerations made at the beginning of this section permit us to replace the upper limit by m in order to extract the logarithmic term which arises from the infrared behaviour:

$$\int \frac{d^3 l}{(2\pi)^3} \frac{i}{4m} \frac{1}{|\mathbf{l}|^2 + \gamma^2} \frac{m\pi}{|\mathbf{l}|} \arctan \frac{|\mathbf{l}|}{\gamma} \approx \frac{i}{16} \ln \frac{1}{\alpha}, \quad (57)$$

where we have approximated $\arctan \frac{|\mathbf{l}|}{\gamma}$ to $\frac{\pi}{2}$.

Collecting all the factors, we have as a contribution to $G_{(R)}^{(\lambda)\rho}$:

$$\frac{e^3}{(4\pi)^2} \frac{m^2}{4} \ln \frac{1}{\alpha} \times \sum_{\sigma \in S_3} \text{Tr} \{ \mathcal{O}_{\sigma(234)}^{(\lambda)} (1 + \gamma_0) \gamma_\rho (1 - \gamma_0) \}, \quad (58)$$

where σ is a permutation of (234). The amplitude receives a contribution:

$$\Delta M_{AR}^{(m\lambda)} = \alpha^2 \log \alpha \frac{1}{16} \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \{ \Psi^{(m)}(p) \gamma_\rho \} \times \sum_{\sigma \in S_3} \text{Tr} \{ \mathcal{O}_{\sigma(234)}^{(\lambda)} (1 + \gamma_0) \gamma_\rho (1 - \gamma_0) \}, \quad (59)$$

and the corresponding correction to the width is the integral over the three final photon phase space of

$$\frac{1}{3} \sum_m \frac{1}{3!} \sum_\lambda 2 \text{Re} (\Delta M_{AR}^{(m\lambda)} * M_0^{(m\lambda)}). \quad (60)$$

The lowest order amplitude, M_0 , can be written in our notation as

$$M_0^{(m,\lambda)} = \sum_{\sigma \in S_3} \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left\{ \Psi^{(m)}(p) \mathcal{O}_{\sigma(234)}^{(\lambda)} \right\}. \quad (61)$$

Using techniques similar to those described in [5], it is possible to rewrite the product of the traces in such a way that the leading width Γ_0 appears explicitly. We find

$$\Delta \Gamma_{AR} = -\alpha^2 \ln \frac{1}{\alpha} \Gamma_0. \quad (62)$$

This result is in agreement with previous calculations [9, 15, 12].

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Appendix A

In this appendix we discuss the contribution to M_{AB} of the term $tr_{\mu\nu\rho}(k) - tr_{\mu\nu\rho}(0)$ and of the vertex counterterm, using as a regulator a sharp cutoff Λ in momentum space. Of course the physical prediction coincides with that obtained in Sect. 2 in dimensional regularization.

As discussed in Sect. 2, the part of the integral (8) given by $tr_{\mu\nu\rho}(k) - tr_{\mu\nu\rho}(0)$ has no infrared singularity and we can put the internal positronium momentum p equal to zero in the loop integral, making only an error of order α^2 . Hence the integral in p can be trivially performed; it remains the loop integral. Introducing Feynman parameters, making simple integrations in⁵

$$-i e \frac{\alpha}{4\pi} \int \frac{d^4 k}{i\pi^2} \times \frac{\gamma_\mu \hat{k} \gamma_\rho \hat{k} \gamma_\nu \Delta_{\mu\nu}(k)}{k^2 \left[(k + \frac{1}{2}P)^2 - m^2 \right] \left[(k - \frac{1}{2}P)^2 - m^2 \right]} \quad (A.1)$$

and ignoring a term proportional to P_ρ , which does not contribute to the amplitude M_{AB} due to the gauge invariance of the light-light scattering tensor, we get

$$-i e \frac{\alpha}{4\pi} \left(3 \ln \frac{\Lambda^2}{m^2} + \frac{5}{2} \right) \gamma_\rho. \quad (A.2)$$

Therefore the contribution of this term to the amplitude is:

$$\frac{\alpha}{4\pi} \left(3 \ln \frac{\Lambda^2}{m^2} + \frac{5}{2} \right) \left(\frac{\alpha}{\pi} M_A \right). \quad (A.3)$$

The computation of the self-energy – to get the vertex counterterm – is more subtle. The mass operator to lowest order is:

$$\Sigma(l) = -i \frac{\alpha}{4\pi} \int \frac{d^4 k}{i\pi^2} \frac{\gamma_\nu (\hat{l} - \hat{k} + m) \gamma_\mu \Delta_{\mu\nu}(k)}{(k^2 - \lambda^2) [(l-k)^2 - m^2]}, \quad (A.4)$$

⁵ In this case $\Delta_{\mu\nu}(k)$ is given by (11)

where λ is the fictitious photon mass. Note that in the FY gauge the numerator does not contain any term which produces linear divergences, therefore the Feynman trick to join the denominators and the subsequent shift of the loop momentum can be used, giving the result:

$$\Sigma(p) = \frac{-i\alpha}{2\pi} \int_0^1 dx \left[(3m - p(1+x)) \left(\ln \frac{\Lambda^2}{a^2} - 1 \right) - \frac{p}{2a^2} \left((1-x)a^2 + 4x^2(1-x)p^2 \right) \right], \quad (A.5)$$

with

$$\begin{aligned} a^2 &= m^2 (x^2 + \nu(1-x) + x(1-x)\rho), \\ \nu &= \frac{\lambda^2}{m^2}, \\ \rho &= 1 - \frac{p^2}{m^2}. \end{aligned} \quad (A.6)$$

The electron wave function renormalization constant, in the on-mass-shell renormalization scheme, is defined as

$$Z = 1 + \left. \frac{d\Sigma(p)}{dp} \right|_{\hat{p}=m, p^2=m^2}. \quad (A.7)$$

Simple algebra leads to the following expression for Z_{FY}^A :

$$1 + \frac{\alpha}{2\pi} \int_0^1 dx \left[-(1+x) \left(\ln \frac{\Lambda^2}{m^2} - 2 \ln x - 1 \right) - \frac{9}{2} (1-x) + 2J(x, \nu, \rho) \right] \Bigg|_{\nu, \rho \rightarrow 0}, \quad (A.8)$$

where

$$J(x, \nu, \rho) = \frac{x(1-x)^2(2-x)(\nu + \rho x)}{[x^2 + \nu(1-x) + x(1-x)\rho]^2}. \quad (A.9)$$

The quantity $\int_0^1 J dx$ depends on the way in which ν and ρ tend to zero:

$$\begin{aligned} \int_0^1 J dx &= \frac{1}{2} \rho \ll \nu \rightarrow 0, \\ \int_0^1 J dx &= 2\nu \ll \rho \rightarrow 0. \end{aligned} \quad (A.10)$$

Y. Tomozawa [18] showed that the right result appears in the limit $\nu \ll \rho \rightarrow 0$. The resulting expression is then:

$$Z_{FY}^A = 1 - \frac{\alpha}{4\pi} \left(3 \ln \frac{\Lambda^2}{m^2} - \frac{3}{2} \right). \quad (A.11)$$

This implies that the contribution of the vertex counterterm insertion is:

$$-\frac{\alpha}{4\pi} \left(3 \ln \frac{\Lambda^2}{m^2} - \frac{3}{2} \right) \left(\frac{\alpha}{\pi} M_A \right). \quad (A.12)$$

To have the total expression of M_{AB} we only need now the contribution of the term $tr_{\mu\nu\rho}(0)$. It does not contain any divergence, hence it can be recovered directly by (16). We have:

$$\left(1 - 3 \frac{\alpha}{\pi} \right) \left(\frac{\alpha}{\pi} M_A \right). \quad (A.13)$$

Collecting the results of (A.13), (A.3) and (A.12), we get the same as in dimensional regularization (28).

Appendix B

We write here the explicit expressions for the quantity $R(234)$ entering the tensor G . Accommodation from the results of paper [24] to the annihilation channel was done in the paper [23].

For the case $\epsilon = m$ (notation of [23]) we obtain:

$$\begin{aligned}
\mathcal{E}_{+++}^{(1)}(234) &= \frac{2a_3a_4}{\nu_3} + \frac{2a_3}{\nu_3} \left(\frac{2a_3a_4}{a_2} + \frac{2a_4}{\nu_3} - a_3 \right) (B(a_3) - B(1)) \\
&+ 2a_3a_4 \left(\frac{2}{a_2} + \frac{1}{\nu_4} \right) (B(a_4) - B(1)) \\
&+ \frac{2a_3}{a_2} \left(a_3 - a_4 - \frac{2a_3a_4}{a_2} \right) \\
&\times (T(a_3) + T(a_4) - T(1) - I(a_3, a_4)) \\
&- \frac{a_3}{a_4} T(a_2) + \frac{a_3}{a_2} (T(a_2) - T(a_4)) - T(a_3) \\
&+ a_3 \left(\frac{3}{\nu_3} - \frac{2a_4}{\nu_3^2} - \frac{1}{a_2} - \frac{1}{a_4} \right) (T(a_3) - T(1)) \\
&+ \frac{\nu_2(a_2 - a_4)}{a_2a_4} I(a_2, a_3) - \frac{\nu_2a_3}{a_2a_4} I(a_2, a_4) \\
&- \frac{a_3}{\nu_4} (T(a_4) - T(1)) \\
&+ \left(2 - \frac{a_3}{a_4} + \frac{3a_3}{a_2} \right) I(a_3, a_4); \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{-++}^{(1)}(234) &= a_3 \left(\frac{1}{a_4} - \frac{1}{a_2} \right) (T(a_2) + T(a_3) + T(a_4) - T(1)) \\
&- \frac{\nu_2}{a_4} I(a_2, a_3) + \frac{\nu_4}{a_2} I(a_3, a_4); \tag{B.2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{+++}^{(2)}(234) &= \left(\frac{4a_3}{a_2} - \frac{2a_3}{\nu_3} \right) (B(a_3) - B(1)) \\
&+ \left(\frac{4a_4}{a_2} - \frac{2a_4}{\nu_4} \right) (B(a_4) - B(1)) \\
&- \left(\frac{4a_3a_4}{a_2^2} + \frac{2\nu_2}{a_2} \right) (T(a_3) + T(a_4) - T(1)) \\
&- I(a_3, a_4) \left(\frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right) T(a_2) \\
&- \left(\frac{a_2}{\nu_3a_4} + \frac{3}{a_2} \right) T(a_3) - \left(\frac{a_2}{\nu_4a_3} + \frac{3}{a_2} \right) T(a_4) \\
&+ \left(\frac{\nu_2}{a_3a_4} - \frac{1}{\nu_3} - \frac{1}{\nu_4} + \frac{3}{a_2} \right) T(1) \\
&+ \left(\frac{a_4 + 1}{a_3a_2} + \frac{\nu_3}{a_2a_4} \right) I(a_2, a_3) \\
&+ \left(\frac{a_3 + 1}{a_4a_2} + \frac{\nu_4}{a_2a_3} \right) I(a_2, a_4) \\
&+ \left(\frac{\nu_2 + 1}{a_4a_3} + \frac{5}{a_2} \right) I(a_3, a_4);
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{-++}^{(2)}(234) &= -2 - \left(\frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right) \\
&\times (T(a_2) + T(a_3) + T(a_4) - T(1)) \\
&+ \left(\frac{1}{a_4} + \frac{1}{a_2a_3} \right) I(a_2, a_3) + \left(\frac{1}{a_3} + \frac{1}{a_2a_4} \right) I(a_2, a_4) \\
&+ \left(\frac{1}{a_2} + \frac{1}{a_4a_3} \right) I(a_3, a_4). \tag{B.3}
\end{aligned}$$

In the previous formulas we used the notations:

$$a_i = 1 - \nu_i, \quad \nu_2 + \nu_3 + \nu_4 = 2, \tag{B.4}$$

$$B(z) = -1 + \sqrt{\frac{1}{z} - 1} \arcsin \sqrt{z}, \tag{B.5}$$

$$T(z) = -(\arcsin \sqrt{z})^2, \tag{B.6}$$

$$I(a_3, a_2) = F(a_3, \gamma) + F(a_2, \gamma) - F(1, \gamma), \tag{B.7}$$

$$F(a, \gamma) = \int_0^1 \frac{dx}{\gamma^2 - x^2} \ln [1 - a(1 - x^2)], \tag{B.8}$$

$$\text{with } \gamma = \sqrt{1 + \frac{a_4}{a_3a_2}}.$$

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